7.3. Substituição Trigonométrica

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$$\int \frac{1}{\sqrt{9x^2+6x-8}}\,dx = \int \frac{1}{\sqrt{(3x+1)^2-9}}\,dx$$

$$= \int \frac{1}{3\cancel{tg\theta}} \cdot \sec\theta\,\cancel{tg\theta}\,d\theta = \frac{1}{3}\int \sec\theta\,d\theta$$

$$= \frac{1}{3}\ln|\sec\theta + tg\theta| + c'$$

$3x+1 = 3\sec\theta$

$\cancel{3}\,dx = \cancel{3}\,\sec\theta\,tg\theta\,d\theta$

$\sqrt{(3x+1)^2-9} = \sqrt{9\sec^2\theta - 9} = 3\sqrt{\sec^2\theta - 1} = 3\sqrt{tg^2\theta} = 3\,tg\theta$

$$= \frac{1}{3}\cdot \ln\left|\frac{3x+1}{3} + \frac{\sqrt{(3x+1)^2-9}}{3}\right| + c'$$

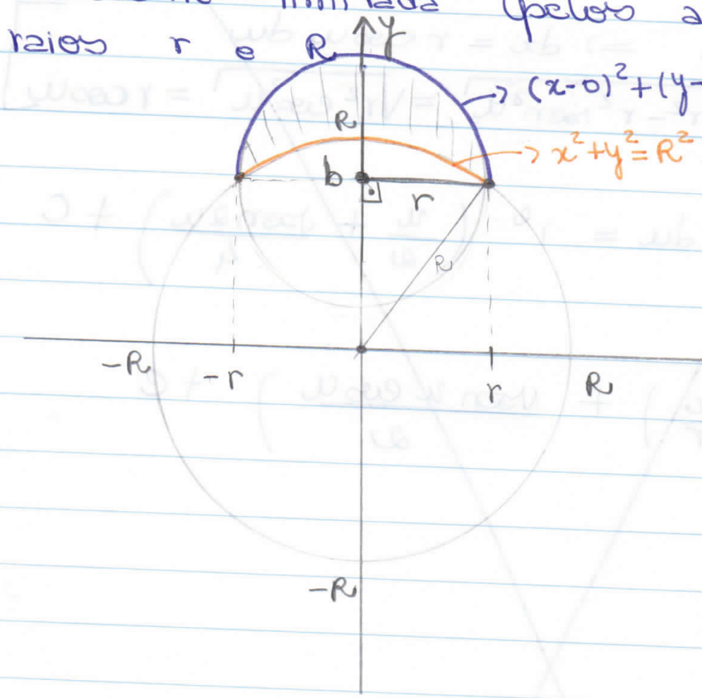
$\sec\theta = \dfrac{3x+1}{3}$

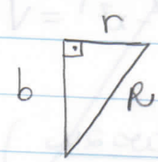
$tg\theta = \dfrac{\sqrt{(3x+1)^2-9}}{3}$

$$= \frac{1}{3}\ln\left|3x+1+\sqrt{(3x+1)^2-9}\right| - \ln 3 + c' \qquad = c$$

7.3

(39) Encontre a área da região em forma de lua crescente limitada pelos arcos dos círculos de raios r e R





$b^2+r^2=R^2$

$b^2=R^2-r^2$

$b=\pm\sqrt{R^2-r^2}$

$b=\sqrt{R^2-r^2}$

$b>0$

Círculo de raio r : $x^2+(y-\sqrt{R^2-r^2})^2=r^2$

Círculo de raio R : $x^2+y^2=R^2$

$\begin{pmatrix}\text{Raio } r\\ y>0\end{pmatrix}$: $x^2+(y-b)^2=r^2 \Rightarrow (y-b)^2=r^2-x^2$

$y-b=\sqrt{r^2-x^2} \Rightarrow y=\sqrt{r^2-x^2}+b$

$\begin{pmatrix}\text{Raio } R\\ y>0\end{pmatrix}$: $x^2+y^2=R^2 \Rightarrow y^2=R^2-x^2$

$y=\sqrt{R^2-x^2}$

$$A=\int_{-r}^{r}(\sqrt{r^2-x^2}+b)\,dx-\int_{-r}^{r}\sqrt{R^2-x^2}\,dx$$

$$A=\int_{-r}^{r}\sqrt{r^2-x^2}\,dx+b\int_{-r}^{r}dx-\int_{-r}^{r}\sqrt{R^2-x^2}\,dx$$

39 cont...

$$A = \int_{-r}^{r} \left(\sqrt{r^2-x^2}+b\right) - \left(\sqrt{R^2-x^2}\right) dx$$

- $\int \sqrt{r^2-x^2}\, dx = \int r\cos\theta \cdot r\cos\theta\, d\theta = r^2 \int \cos^2\theta\, d\theta$

$x = r\,\mathrm{sen}\,\theta$

$dx = r\cos\theta\, d\theta$

$$\sqrt{r^2-x^2} = \sqrt{r^2 - r^2\mathrm{sen}^2\theta} = \sqrt{r^2\cos^2\theta} = r\cos\theta$$

$$= r^2 \int \frac{1+\cos 2\theta}{2}\, d\theta = \frac{r^2}{2}\left(\int d\theta + \int \cos 2\theta\, d\theta\right) =$$

$$= \frac{r^2}{2}\left(\theta + \frac{\mathrm{sen}\,2\theta}{2}\right) + C = \frac{r^2}{2}\left(\theta + \frac{1}{\cancel{2}}\cdot \cancel{2}\,\mathrm{sen}\,\theta\cos\theta\right) + C$$

$$= \frac{r^2}{2}\left(\mathrm{sen}^{-1}\left(\frac{x}{r}\right) + \frac{x}{r}\cdot\frac{\sqrt{r^2-x^2}}{r}\right) + C$$

$\mathrm{sen}\,\theta = \frac{x}{r} \iff \theta = \mathrm{sen}^{-1}\left(\frac{x}{r}\right)$

$\cos\theta = \frac{\sqrt{r^2-x^2}}{r}$

$$= \frac{r^2}{2}\,\mathrm{sen}^{-1}\left(\frac{x}{r}\right) + \frac{\cancel{r^2}}{2}\cdot\frac{x}{\cancel{r^2}}\sqrt{r^2-x^2} + C$$

$$= \frac{r^2}{2}\,\mathrm{sen}^{-1}\left(\frac{x}{r}\right) + \frac{x}{2}\sqrt{r^2-x^2} + C$$

- $$\int \sqrt{R^2 - x^2}\, dx = \frac{R^2}{2} \operatorname{sen}^{-1}\left(\frac{x}{R}\right) + \frac{x}{2}\sqrt{R^2 - x^2} + C$$

Assim;

$$A = \int_{-r}^{r} \sqrt{r^2 - x^2}\, dx + b \int_{-r}^{r} dx - \int_{-r}^{r} \sqrt{R^2 - x^2}\, dx$$

$$A = \left(\frac{r^2}{2} \operatorname{sen}^{-1}\left(\frac{x}{r}\right) + \frac{x}{2}\sqrt{r^2 - x^2} \right)\Bigg]_{-r}^{r} + b\,x\Big]_{-r}^{r} - \left(\frac{R^2}{2} \operatorname{sen}^{-1}\left(\frac{x}{R}\right) + \frac{x}{2}\sqrt{R^2 - x^2} \right)\Bigg]_{-r}^{r}$$

$$A = \frac{r^2}{2} \underbrace{\operatorname{sen}^{-1}\left(\frac{r}{r}\right)}_{=\frac{\pi}{2}} + \frac{r}{2}\cancelto{0}{\sqrt{r^2 - r^2}} - \left(\frac{r^2}{2} \operatorname{sen}^{-1}\left(\frac{-r}{r}\right) - \frac{r}{2}\cancelto{0}{\sqrt{r^2 - \underbrace{(-r)^2}_{r^2}}} \right)$$

$$+ br - b(-r) - \left\{ \frac{R^2}{2} \operatorname{sen}^{-1}\left(\frac{r}{R}\right) + \frac{r}{2}\sqrt{R^2 - r^2} - \frac{R^2}{2} \underbrace{\operatorname{sen}^{-1}\left(\frac{-r}{R}\right)}_{= -\operatorname{sen}^{-1}\left(\frac{r}{R}\right)} + \frac{r}{2}\sqrt{R^2 - (-r)^2} \right\}$$

$$= \frac{r^2}{2} \cdot \frac{\pi}{2} - \frac{r^2}{2}\left(-\frac{\pi}{2}\right) + 2br - \frac{R^2}{2} \operatorname{sen}^{-1}\left(\frac{r}{R}\right) - \frac{r}{2}\sqrt{R^2 - r^2}$$

$$+ \frac{R^2}{2}\left(- \operatorname{sen}^{-1}\left(\frac{r}{R}\right)\right) - \frac{r}{2}\sqrt{R^2 - r^2}$$

$$= \frac{r^2}{4}\pi + \frac{r^2}{4}\pi + 2r\sqrt{R^2 - r^2} - \frac{R^2}{2} \operatorname{sen}^{-1}\left(\frac{r}{R}\right) - r\sqrt{R^2 - r^2}$$

$$- \frac{R^2}{2} \operatorname{sen}^{-1}\left(\frac{r}{R}\right) = \frac{r^2\pi}{2} + r\sqrt{R^2 - r^2} - R^2 \operatorname{sen}^{-1}\left(\frac{r}{R}\right)$$

7.8. Integrais Impróprias

56 Avalie $\displaystyle\int_2^{\infty} \frac{1}{x\sqrt{x^2-4}}\,dx$

A integral acima é imprópria por duas razões: o intervalo $[2,\infty)$ é infinito e o integrando tem uma descontinuidade infinita em 2 $\left(\frac{1}{2\sqrt{2^2-4}} = \text{"}\frac{1}{0}\text{"}\right)$

Devemos então separar a integral em duas partes separando esses dois problemas.

$$\int_2^{\infty} \frac{1}{x\sqrt{x^2-4}}\,dx = \int_2^{3} \frac{dx}{x\sqrt{x^2-4}} + \int_3^{+\infty} \frac{dx}{x\sqrt{x^2-4}}$$

- $$\int \frac{dx}{x\sqrt{x^2-4}} \underset{\downarrow}{=} \int \frac{\cancel{2}\,\cancel{\sec\theta}\,\cancel{tg\theta}}{\cancel{2}\,\cancel{\sec\theta}\cdot 2\,\cancel{tg\theta}}\,d\theta = \frac{1}{2}\int d\theta = \frac{\theta}{2} + c$$

$$\boxed{\begin{gathered} x = 2\sec\theta \;;\; 0 \leq \theta < \frac{\pi}{2} \text{ ou } \pi \leq \theta < \frac{3\pi}{2} \\ dx = 2\sec\theta\, tg\theta \\ \sqrt{x^2-4} = \sqrt{4\sec^2\theta - 4} = 2\sqrt{\sec^2\theta - 1} = 2\sqrt{tg^2\theta} = 2tg\theta \end{gathered}}$$

$$\underset{\downarrow}{=} \frac{1}{2}\cdot \sec^{-1}\left(\frac{x}{2}\right) + c$$

$$\sec\theta = \frac{x}{2}$$

$$\Rightarrow \theta = \sec^{-1}\left(\frac{x}{2}\right)$$

- $$\int_2^3 \frac{dx}{x\sqrt{x^2-4}} = \lim_{t \to 2^+} \int_t^3 \frac{dx}{x\sqrt{x^2-4}} = \lim_{t \to 2^+} \left(\frac{1}{2} \cdot \sec^{-1}\left(\frac{x}{2}\right)\right)\Bigg]_t^3$$

$\sec^{-1}(1) = 0$, pois $\sec(0)=1$

$$= \lim_{t \to 2^+} \left(\frac{1}{2}\sec^{-1}\left(\frac{3}{2}\right) - \cancel{\frac{1}{2}\sec^{-1}\left(\frac{t}{2}\right)}\right) = \frac{1}{2}\sec^{-1}\left(\frac{3}{2}\right)$$

- $$\int_3^\infty \frac{dx}{x\sqrt{x^2-4}} = \lim_{t \to +\infty} \int_3^t \frac{dx}{x\sqrt{x^2-4}} = \lim_{t \to +\infty} \left(\frac{1}{2} \cdot \sec^{-1}\left(\frac{x}{2}\right)\right)\Bigg]_3^t$$

$$= \lim_{t \to +\infty} \left(\frac{1}{2}\sec^{-1}\left(\frac{t}{2}\right) - \frac{1}{2}\sec^{-1}\left(\frac{3}{2}\right)\right) = \frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2}\sec^{-1}\left(\frac{3}{2}\right)$$

$$(*) \lim_{t \to +\infty} \sec^{-1}\left(\frac{t}{2}\right) = \frac{\pi}{2}$$

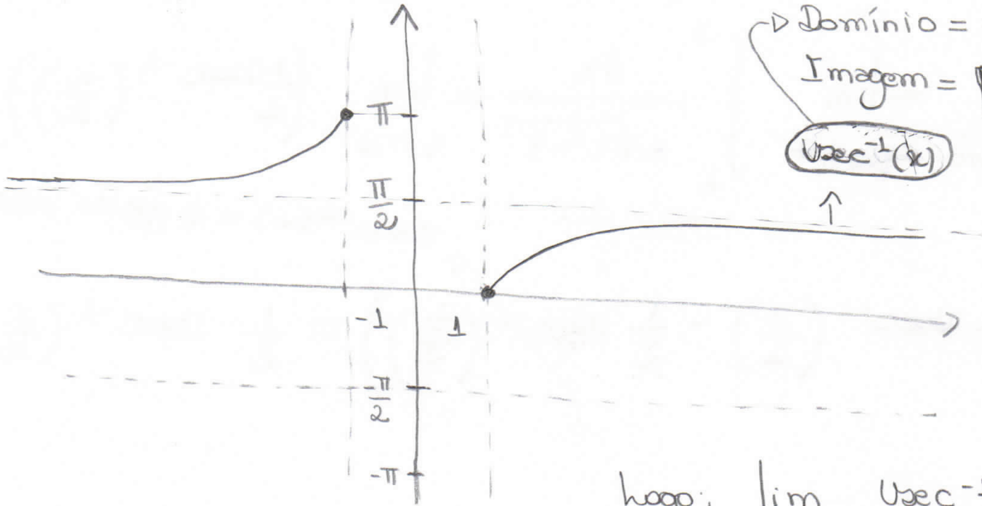
Logo,

$$\int_2^\infty \frac{dx}{x\sqrt{x^2-4}} = \int_2^3 \frac{dx}{x\sqrt{x^2-4}} + \int_3^\infty \frac{dx}{x\sqrt{x^2-4}} =$$

$$= \cancel{\frac{1}{2}\sec^{-1}\left(\frac{3}{2}\right)} + \frac{\pi}{4} - \cancel{\frac{1}{2}\sec^{-1}\left(\frac{3}{2}\right)} = \frac{\pi}{4}$$

//

(*)



Domínio $= (-\infty, -1] \cup [1, +\infty)$

Imagem $= \left[0, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

$\sec^{-1}(x)$

logo, $\lim_{x \to +\infty} \sec^{-1} x = \frac{\pi}{2}$

77 Calcule c para o qual a integral

$$\int_0^{\infty} \underbrace{\left(\frac{1}{\sqrt{x^2+4}} - \frac{c}{x+2}\right)}_{f(x)} dx$$

Obs: $[0, +\infty) \subset \text{Dom} f$

converge. Avalie a integral para esse valor de c.

$$\int_0^{\infty} \left(\frac{1}{\sqrt{x^2+4}} - \frac{c}{x+2}\right) dx = \lim_{t \to \infty} \int_0^{t} \left(\frac{1}{\sqrt{x^2+4}} - \frac{c}{x+2}\right) dx$$

- $$\int \left(\frac{1}{\sqrt{x^2+4}} - \frac{c}{x+2}\right) dx = \int \frac{dx}{\sqrt{x^2+4}} - c \cdot \int \frac{dx}{x+2} =$$

$$= \ln\left|\sqrt{x^2+4} + x\right| - c \cdot \ln|x+2| + k$$

pois

$$\int \frac{dx}{\sqrt{x^2+4}} \underset{\downarrow}{=} \int \frac{\cancel{2}\sec^{\cancel{2}} u \, du}{\cancel{2}\cancel{\sec u}} = \int \sec u \, du = \ln|\sec u + \operatorname{tg} u| + k'$$

$x = 2 \operatorname{tg} u$

$dx = 2 \sec^2 u \, du$

$$\sqrt{x^2+4} = \sqrt{4\operatorname{tg}^2 u + 4} = \sqrt{4(\operatorname{tg}^2 u + 1)} = 2\sqrt{\sec^2 u} = 2 \sec u$$

$$\int \frac{dx}{\sqrt{x^2+4}} = \ln|\sec u + tg\,u| + k' = \ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + k'$$

$$\sec u = \frac{\sqrt{x^2+4}}{2} \quad (\sqrt{x^2+4} = 2\sec u)$$

$$tg\,u = \frac{x}{2} \quad (x = 2\,tg\,u)$$

$$= \ln\left|\frac{\sqrt{x^2+4}+x}{2}\right| + k' = \ln|\sqrt{x^2+4}+x| \underbrace{- \ln|2| + k'}_{= k}$$

$$= \ln|\sqrt{x^2+4}+x| + k$$

$$\int_0^\infty \left(\frac{1}{\sqrt{x^2+4}} - \frac{c}{x+2}\right)dx = \lim_{t\to+\infty} \int_0^t \left(\frac{1}{\sqrt{x^2+4}} - \frac{c}{x+2}\right)dx$$

$$= \lim_{t\to+\infty} \left(\ln|\sqrt{x^2+4}+x| - c\ln|x+2|\right)\Big]_0^t$$

$$= \lim_{t\to+\infty} \left(\ln|\sqrt{x^2+4}+x| - \ln|x+2|^c\right)\Big]_0^t$$

$$= \lim_{t\to+\infty} \left(\ln\left|\frac{\sqrt{x^2+4}+x}{(x+2)^c}\right|\right)\Big]_0^t$$

$$= \lim_{t\to+\infty} \left(\ln\left|\frac{\sqrt{t^2+4}+t}{(t+2)^c}\right| - \ln\left|\frac{\sqrt{0^2+4}+0}{(0+2)^c}\right|\right)$$

$$= \lim_{t\to+\infty} \left(\ln\left|\frac{\sqrt{t^2+4}+t}{(t+2)^c}\right| - \ln\left|\overbrace{\frac{2}{2^c}}^{\frac{1}{2^{c-1}} = 2^{-(c-1)}}\right|\right)$$