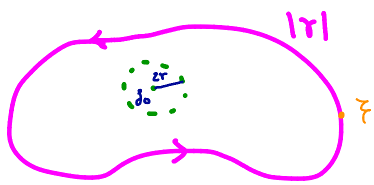
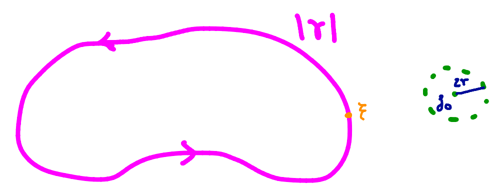
Lema 5.9: Seja γ um caminho fechado suave por partes em $\mathbb{C}$. Para cada $h: |\gamma| \to \mathbb{C}$ contínua e cada inteiro $n \geq 1$, definimos $\tilde{h}_n: \mathbb{C}\setminus|\gamma| \to \mathbb{C}$ por

$$\tilde{h}_n(z) = \int_\gamma \frac{h(\zeta)}{(\zeta - z)^n}\, d\zeta$$

Então cada $\tilde{h}_n$ é analítica e $(\tilde{h}_n)'(z) = n\, \tilde{h}_{n+1}(z)$, $\forall z \in \mathbb{C}\setminus|\gamma|$.

<u>dem:</u> Fixe $z_0 \in \mathbb{C}\setminus|\gamma|$ e tome $r>0$ tal que $\Delta(z_0, 2r) \subset \mathbb{C}\setminus|\gamma|$





Assim, se $\zeta \in |\gamma|$ temos que $|\zeta - z_0| > 2r$. (*)

Afirmação 1: Toda $\tilde{h}_n$ é limitada em $\Delta(z_0, r)$

Compacto

h contínua em $|\gamma|$

$$|\tilde{h}_n(z)| = \left|\int_\gamma \frac{h(\zeta)}{(\zeta - z)^n}\, d\zeta\right| \leq \int_\gamma \frac{|h(\zeta)|}{|\zeta - z|^n}\, |d\zeta| \leq \frac{1}{r^n}\int_\gamma |h(\zeta)|\, |d\zeta| \leq \frac{1}{r^n}\cdot M, \ \forall z \in \Delta(z_0, r)$$

$$\boxed{|\zeta - z| = |\zeta - z_0 + z_0 - z| \geq |\zeta - z_0| - |z - z_0| \underset{(*)}{>} 2r - r = r, \ \forall z \in \Delta(z_0, r)}$$

Fixando $h: |\gamma| \to \mathbb{C}$ contínua, defina $g: |\gamma| \to \mathbb{C}$ por

$$\boxed{g(\zeta) = \frac{h(\zeta)}{\zeta - z_0}}$$ (note que g é contínua)

Afirmação 2: Toda $\tilde{h}_n$ é contínua em z_0

Vamos provar esta afirmação usando indução sobre n.

(1º) $\tilde{h}_1$ é contínua em z_0

$$\tilde{h}_1(z) = \int_\gamma \frac{h(\xi)}{(\xi - z)}\,d\xi = \int_\gamma \frac{h(\xi)}{(\xi - z)}\frac{(\xi - z_0)}{(\xi - z_0)}\,d\xi = \int_\gamma \overbrace{\left[\frac{h(\xi)}{\xi - z_0}\right]}^{g(\xi)} \cdot \left[\frac{\xi - z_0}{\xi - z}\right] d\xi = \int_\gamma g(\xi)\left(\frac{\xi - z_0}{\xi - z}\right) d\xi$$

$$\tilde{h}_1(z_0) = \int_\gamma \underbrace{\frac{h(\xi)}{\xi - z_0}}_{g(\xi)}\,d\xi = \int_\gamma g(\xi)\,d\xi$$

Assim;

$$\tilde{h}_1(z) - \tilde{h}_1(z_0) = \int_\gamma g(\xi)\left(\frac{\xi - z_0}{\xi - z}\right) d\xi - \int_\gamma g(\xi)\,d\xi = \int_\gamma g(\xi)\left[\frac{\xi - z_0}{\xi - z} - 1\right] d\xi$$

$$= \int_\gamma g(\xi)\left[\frac{\xi - z_0 - \xi + z}{\xi - z}\right] d\xi = \int_\gamma g(\xi)\,\frac{z - z_0}{\xi - z}\,d\xi$$

$$= (z - z_0)\underbrace{\int_\gamma \frac{g(\xi)}{\xi - z}\,d\xi}_{\tilde{g}_1(z)} = \underbrace{(z - z_0)}_{\downarrow\, 0}\ \underbrace{\tilde{g}_1(z)}_{\text{limitado em } D(z_0, r)} \xrightarrow{z \to z_0} 0$$

$\tilde{g}_1(z)$ (usamos a definição de $\tilde{h}_n$ para $h = g$ e $n = 1$)

Logo; $\lim\limits_{z \to z_0} \tilde{h}_1(z) = \tilde{h}_1(z_0) \Rightarrow \tilde{h}_1$ é contínua em z_0.

(2º) Suponhamos que os $\tilde{h}_m$ são contínuas em z_0, vamos mostrar que $\tilde{h}_{m+1}$ é contínua em z_0. $(m \geq 1)$

$$\tilde{h}_{m+1}(z) = \int_\gamma \frac{h(\xi)}{(\xi - z)^{m+1}}\,d\xi = \int_\gamma \frac{h(\xi)}{(\xi - z)^{m+1}}\frac{(\xi - z_0)}{(\xi - z_0)}\,d\xi = \int_\gamma \frac{h(\xi)}{(\xi - z)^{m+1}}\frac{(\xi - z + z - z_0)}{(\xi - z_0)}\,d\xi =$$

$$= \int_\gamma \underbrace{\frac{h(\xi)}{\xi - z_0}}_{g(\xi)}\left[\frac{\xi - z + z - z_0}{(\xi - z)^{m+1}}\right] d\xi = \int_\gamma g(\xi)\left[\frac{\xi - z}{(\xi - z)^{m+1}} + \frac{z - z_0}{(\xi - z)^{m+1}}\right] d\xi$$

$$= \int_\gamma \left[\frac{g(\xi)}{(\xi - z)^m} + \frac{g(\xi)}{(\xi - z)^{m+1}}(z - z_0)\right] d\xi = \tilde{g}_m(z) + \tilde{g}_{m+1}(z)\,(z - z_0)$$

$$\tilde{h}_{m+1}(z_0) = \int_\gamma \frac{h(\xi)}{(\xi - z_0)^m}\, d\xi = \int_\gamma \frac{h(\xi)}{(\xi - z_0)} \cdot \frac{1}{(\xi - z_0)^m}\, d\xi = \int_\gamma \frac{g(\xi)}{(\xi - z_0)^m}\, d\xi = \tilde{g}_m(z_0)$$

Assim;

$$\tilde{h}_{m+1}(z) - \tilde{h}_{m+1}(z_0) = \tilde{g}_m(z) + \tilde{g}_{m+1}(z)\,(z - z_0) - \tilde{g}_m(z_0)$$

$$= \underbrace{\tilde{g}_m(z) - \tilde{g}_m(z_0)}_{0} + \underbrace{\tilde{g}_{m+1}(z)}_{\text{limitada em } \Delta(z_0, r)}\, \underbrace{(z - z_0)}_{0} \xrightarrow{z \to z_0} 0$$

pois pela hip de indução $\tilde{g}_m$ é contínua em z_0

Logo; $\lim\limits_{z \to z_0} \tilde{h}_{m+1}(z) = \tilde{h}_{m+1}(z_0) \Longrightarrow \tilde{h}_{m+1}$ é contínua em z_0.

Afirmação 3: Toda $\tilde{h}_n$ é diferenciável em z_0 e $(\tilde{h}_n)'(z_0) = n\, \tilde{h}_{n+1}(z_0)$

Vamos provar esta afirmação usando indução sobre n.

(1º) $\tilde{h}_1$ é diferenciável em z_0

$$\underbrace{\frac{\tilde{h}_1(z) - \tilde{h}_1(z_0)}{z - z_0}}_{(\tilde{h}_1)'(z_0)} = \frac{\tilde{g}_1(z)\,(z - z_0)}{z - z_0} = \underbrace{\tilde{g}_1(z)}_{\tilde{g}_1(z_0)}, \quad \forall z \in \Delta^*(z_0, r)$$

$z \to z_0$

$$\tilde{g}_1(z_0) = \int_\gamma \frac{\frac{h(\xi)}{(\xi - z_0)}}{(\xi - z_0)}\, d\xi = \int_\gamma \frac{h(\xi)}{(\xi - z_0)^2}\, d\xi = \tilde{h}_2(z_0)$$

($\tilde{g}_1$ é cont. em z_0)

Logo, $\tilde{h}_1$ é diferenciável em z_0 e $(\tilde{h}_1)'(z_0) = 1 \cdot \tilde{h}_2(z_0)$.

(2º) Suponhamos que os $\tilde{h}_m$ são dif em z_0, vamos mostrar que $\tilde{h}_{m+1}$ é dif em z_0. $(m \geq 1)$

$$\underbrace{\frac{\tilde{h}_{m+1}(z) - \tilde{h}_{m+1}(z_0)}{z - z_0}}_{\substack{z \to z_0 \\ \downarrow \\ (h_{m+1})'(z_0)}} = \underbrace{\frac{\tilde{g}_m(z) - \tilde{g}_m(z_0)}{z - z_0}}_{\substack{\text{hip de indução} \\ \downarrow \\ \underbrace{(\tilde{g}_m)'(z_0)}_{\substack{\| \\ \underbrace{m\,\tilde{g}_{m+1}(z_0)}_{\substack{\| \\ m\,\tilde{h}_{m+2}(z_0)}}}}}} + \underbrace{\tilde{g}_{m+1}(z)}_{\substack{\downarrow \\ \tilde{g}_{m+1}(z_0) = \tilde{h}_{m+2}(z_0)}} \xrightarrow{z \to z_0} m\,\tilde{h}_{m+2}(z_0) + \tilde{h}_{m+2}(z_0) = (m+1)\,\tilde{h}_{m+2}(z_0)$$

($\tilde{g}_{m+1}$ cont. em z_0)

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